

THE RANK OF THE SEMIGROUP OF TRANSFORMATIONS STABILISING A PARTITION OF A FINITE SET

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ABSTRACT. Let \mathcal{P} be a partition of a finite set X . We say that a full transformation $f : X \rightarrow X$ preserves (or stabilizes) the partition \mathcal{P} if for all $P \in \mathcal{P}$ there exists $Q \in \mathcal{P}$ such that $Pf \subseteq Q$. Let $T(X, \mathcal{P})$ denote the semigroup of all full transformations of X that preserve the partition \mathcal{P} .

In 2005 Huisheng found an upper bound for the minimum size of the generating sets of $T(X, \mathcal{P})$, when \mathcal{P} is a partition in which all of its parts have the same size. In addition, Huisheng conjectured that his bound was exact. In 2009 the first and last authors used representation theory to completely solve Hisheng's conjecture.

The goal of this paper is to solve the much more complex problem of finding the minimum size of the generating sets of $T(X, \mathcal{P})$, when \mathcal{P} is an arbitrary partition. Again we use representation theory to find the minimum number of elements needed to generate the wreath product of finitely many symmetric groups, and then use this result to solve the problem.

The paper ends with a number of problems for experts in group and semigroup theories.

1. INTRODUCTION

If S is a semigroup and U is a subset of S , then we say that U *generates* S if every element of S is expressible as a product of the elements of U . The *rank* of a semigroup S , denoted by $\text{rank } S$, is the least cardinality of a subset that generates S . It is well-known that a finite full transformation semigroup, on at least 3 points, has rank 3, while a finite full partial transformation semigroup, on at least 3 points, has rank 4 (see [17, Exercises 1.9.7 and 1.9.13]). The problem of determining the minimum number of generators of a semigroup is classical, and has been studied extensively; see, for example, [9, 13, 18, 23, 25]. Related notions, such as the *idempotent rank*, the *nilpotent rank* and the relative rank of a subsemigroup, have also been widely investigated; see [5, 10, 11, 12, 14, 15, 16, 19, 22].

In [20], Huisheng posed the problem of finding the rank of the semigroup of transformations preserving a uniform partition (that is, a partition in which all the blocks have equal size). This problem was solved in [7]. In this paper, we solve the general problem of determining the rank of the semigroup of transformations preserving any partition. In the process, we calculate the ranks of some related transformation semigroups. The strategy of the proof is similar to the one used in [7]: we rely on representation theory to find the rank of the group of automorphisms of the partition and then use that result to derive the rank of the semigroup.

Let X be a non-empty finite set, and let \mathcal{P} be a partition of X . A *transformation* is a function from X to itself. We write transformations to the right of their arguments and compose them from left to right. We denote by $T(X, \mathcal{P})$ the semigroup consisting of those transformations f on X such that $(x, y) \in \mathcal{P}$ implies $(xf, yf) \in \mathcal{P}$. The semigroup $T(X, \mathcal{P})$ can be seen as the endomorphism monoid of the relational structure (X, \mathcal{P}) .

We will determine the rank of $T(X, \mathcal{P})$. In order to do this we will determine relative ranks with regard to two subsets of $T(X, \mathcal{P})$. One is the group of units of $T(X, \mathcal{P})$, which is the intersection of $T(X, \mathcal{P})$ with the symmetric group S_X on X ; the other is $\Sigma(X, \mathcal{P})$, consisting of $f \in T(X, \mathcal{P})$ whose image intersects every block of \mathcal{P} . We will denote the group of units of $T(X, \mathcal{P})$ by $S(X, \mathcal{P})$.

The main theorem of this paper is the following.

Theorem 1.1. *Let \mathcal{P} be a partition on X , such that \mathcal{P} has exactly $m_i \geq 2$ blocks of size $n_i \geq 2$, $i = 1, \dots, p$, blocks of unique sizes l_1, \dots, l_q , where $l_i \geq 2$, and t singleton blocks (where p, q, t*

2+1	3	2+1+1	5	2+1+1+1	5	2+1+1+1+1	5	2+1+1+1+1+1	5
		2+2	4	2+2+1	5	2+2+1+1	7	2+2+1+1+1	7
		3+1	5	3+1+1	6	2+2+2	4	2+2+2+1	5
				3+2	5	3+1+1+1	6	3+1+1+1+1	6
				4+1	5	3+2+1	7	3+2+1+1	9
						3+3	4	3+2+2	7
						4+1+1	6	3+3+1	6
						4+2	6	4+1+1+1	6
						5+1	5	4+2+1	8
								4+3	5
								5+1+1	6
								5+2	6
								6+1	5

FIGURE 1. The partitions of 3 to 7 and the ranks of the corresponding monoids.

might be 0). If $|S(X, \mathcal{P})| \geq 3$ then the rank of $T(X, \mathcal{P})$ is

$$\max\{2, 2p + q + g(t)\} + \binom{p+q}{2} + 2p + q + g'(t) - 1 + l + h(p, q, t),$$

where

- $g(0) = g(1) = 0$ and $g(t) = 1$ for $t \geq 2$,
- $g'(0) = 0$ and $g'(t) = 1$ for $t \geq 1$.
- l is the number of values s for which \mathcal{P} has a block of size $s \geq 2$, but no block of size $s - 1$,
- $h(p, q, 0) = 0$, $h(p, q, 1) = p + q$ and $h(p, q, t) = p + q + 1$, if $t \geq 2$.

The rank of $T(X, \mathcal{P})$ is given in Figure 1 for partitions of small values of $|X|$. For comparison, Figure 2 lists the corresponding sizes of the monoids $T(X, \mathcal{P})$.

If U is a subset of a semigroup V , then, as usual we denote the subsemigroup generated by U by $\langle U \rangle$. If U is a subsemigroup of a semigroup V , then the least cardinality of a subset W of V such that $\langle U, W \rangle = V$ is called the *relative rank* of U in V ; this is denoted $\text{rank}(V : U)$.

Since $S(X, \mathcal{P}) \subseteq \Sigma(X, \mathcal{P})$ and the complements of $S(X, \mathcal{P})$ and $\Sigma(X, \mathcal{P})$ are ideals, it follows that

$$\begin{aligned} \text{rank}(T(X, \mathcal{P})) &= \text{rank}(T(X, \mathcal{P}) : \Sigma(X, \mathcal{P})) + \text{rank}(\Sigma(X, \mathcal{P})) \\ &= \text{rank}(T(X, \mathcal{P}) : \Sigma(X, \mathcal{P})) + \text{rank}(\Sigma(X, \mathcal{P}) : S(X, \mathcal{P})) + \text{rank}(S(X, \mathcal{P})) \end{aligned}$$

To prove our main theorem, we will determine that under the given conditions

- $\text{rank}(S(X, \mathcal{P})) = \max\{2, 2p + q + g(t)\}$ (Section 2),
- $\text{rank}(T(X, \mathcal{P}) : \Sigma(X, \mathcal{P})) = \binom{p+q}{2} + p + h(p, q, t)$ (Section 3), and
- $\text{rank}(\Sigma(X, \mathcal{P}) : S(X, \mathcal{P})) = p + q + g'(t) - 1 + l$ (Section 4).

For completeness, we remark that if $S(X, \mathcal{P})$ has two elements, we are in one of the following straightforward cases:

- $|X| = 2$, $\mathcal{P} = \{P_1\}$, $|P_1| = 2$, $\text{rank}(T(X, \mathcal{P})) = 2$.
- $|X| = 2$, $\mathcal{P} = \{P_1, P_2\}$, $|P_1| = |P_2| = 1$, $\text{rank}(T(X, \mathcal{P})) = 2$.
- $|X| = 3$, $\mathcal{P} = \{P_1, P_2\}$, $|P_1| = 2$, $|P_2| = 1$, $\text{rank}(T(X, \mathcal{P})) = 3$.

2+1	6	2+1+1	96	2+1+1+1	875	2+1+1+1+1	10368	2+1+1+1+1+1	151263
		2+2	64	2+2+1	405	2+2+1+1	3600	2+2+1+1+1	41503
		3+1	100	3+1+1	725	2+2+2	1728	2+2+2+1	15379
				3+2	455	3+1+1+1	6480	3+1+1+1+1	74431
				4+1	1285	3+2+1	3024	3+2+1+1	27195
						3+3	2916	3+2+2	12427
						4+1+1	9288	3+3+1	21175
						4+2	5440	4+1+1+1	88837
						5+1	18756	4+2+1	40131
								4+3	30667
								5+1+1	153223
								5+2	91553
								6+1	326599

FIGURE 2. The partitions of 3 to 7 and the sizes of the corresponding monoids.

2. THE RANK OF DIRECT PRODUCTS OF WREATH PRODUCTS OF SYMMETRIC GROUPS

If G and H are permutation groups, then we denote by $G \wr H$ the *wreath product* of G and H . As usual, if $|X| = n$, then we denote the symmetric group S_X on X by S_n ; likewise, in this case, we denote the alternating group by A_n .

Let $n \geq 2$ and let \mathcal{P} be a partition with at least 2 parts. Then we may write $\mathcal{P} = \{P_1, \dots, P_n\}$ such that $|P_i| \leq |P_j|$ when $i < j$, and $i, j \in \{1, \dots, n\}$.

If $f \in T(X, \mathcal{P})$, then we denote by $\bar{f} \in T_n$ the transformation whose action on $\{1, 2, \dots, n\}$ is that induced by the action of f on X/\mathcal{P} . In more details, $(i)\bar{f} = j$ whenever $P_i f \subseteq P_j$. If $f \in S(X, \mathcal{P})$, then it is clear that $(i)\bar{f} = j$ if and only if $|P_i| = |P_j|$.

We start by stating without proof two simple results about $S(X, \mathcal{P})$ and its induced action on $T(X, \mathcal{P})$.

Lemma 2.1. *Let \mathcal{P} be a partition of a set X where the distinct sizes of the blocks are denoted n_i , $i = 1, \dots, k$, and m_i denotes the number of blocks of size n_i . Then the group of units $S(X, \mathcal{P})$ of $T(X, \mathcal{P})$ is isomorphic to*

$$(S_{n_1} \wr S_{m_1}) \times \dots \times (S_{n_k} \wr S_{m_k}).$$

If f is a transformation of a set X , then the *image* of f is the set

$$\text{im}(f) = \{(x)f : x \in X\}$$

and the *kernel* of f is the equivalence relation

$$\ker(f) = \{(x, y) \in X \times X : (x)f = (y)f\},$$

the classes of this relation are referred to as *kernel classes*. If Y is a subset of X , then the *restriction* of f to Y is denoted by $f|_Y$.

Lemma 2.2. *For every block P of \mathcal{P} and $f \in T(X, \mathcal{P})$, let P_f be the multiset of sizes of blocks in the kernel of $f|_P$. For every i, j such that \mathcal{P} has blocks of sizes i and j (not necessarily distinct), let $J_{i,j,f}$ be the multiset of all P_f such that $|P| = i$ and $(P)f$ is contained in a block of size j . Then $g \in S(X, \mathcal{P})fS(X, \mathcal{P})$ if and only if $J_{i,j,f} = J_{i,j,g}$ for all pairs (i, j) .*

For example, let $X = \{1, \dots, 8\}$, $\mathcal{P} = \{P, Q\}$, $P = \{1, 2, 3, 4\}$, $Q = \{5, 6, 7, 8\}$, and define $f \in T(X, \mathcal{P})$ by $(1)f = 2$, $(3)f = 4$, and $(x)f = x$ for $x \neq 1, 3$. Then $P_f = \{2, 2\}$, $Q_f = \{1, 1, 1, 1\}$, and $J_{4,4,f} = \{\{2, 2\}, \{1, 1, 1, 1\}\}$.

If $g \in T(X, \mathcal{P})$ is given by $(1)g = 2$, $(5)g = 6$, and $(x)g = x$ for $x \neq 1, 5$, then $P_g = \{2, 1, 1\} = Q_g$, and $J_{4,4,g} = \{\{2, 1, 1\}, \{2, 1, 1\}\}$. Hence $g \notin S(X, \mathcal{P})fS(X, \mathcal{P})$. Note that f and g have the same multiset of sizes of kernel classes.

We recall also one of the main theorems in [7].

Theorem 2.3. *If X is a finite set such that $|X| \geq 3$ and \mathcal{P} is a uniform partition of X , then $S(X, \mathcal{P})$ is generated by two elements.*

The following lemma is well-known; see, for instance, [21, Lemma 5.3.4].

Lemma 2.4. *The permutation module V of the symmetric group S_n on an n -element set over a field \mathbb{F} of characteristic p has precisely two proper non-trivial submodules:*

$$\begin{aligned} U_1 &= \{(a, a, \dots, a) : a \in \mathbb{F}\} \quad \text{and} \\ U_2 &= \{(a_1, \dots, a_n) : a_1 + \dots + a_n = 0\}. \end{aligned}$$

Furthermore, if p divides n , then $U_1 \leq U_2$; otherwise $V = U_1 \oplus U_2$.

Theorem 2.5. *Let $n_1, \dots, n_k, m_1, \dots, m_k, l_1, \dots, l_u$ be integers such that they are all at least 2 and let*

$$W = (S_{n_1} \wr S_{m_1}) \times \dots \times (S_{n_k} \wr S_{m_k}) \times S_{l_1} \times \dots \times S_{l_u}.$$

If $W \not\cong S_2$, then the rank of W is $\max\{2, 2k + u\}$.

Proof. Let us assume that $W \not\cong C_2$. If $2k + u < 2$, then $k = 0$ and $u = 1$. In this case, $W = S_{l_1}$ is not isomorphic to S_2 , and the rank of W is 2.

Let us show that W cannot be generated by fewer than $2k + u$ elements. Let $i \in \{1, \dots, k\}$. Then $(A_{n_i})^{m_i}$ is a normal subgroup of $S_{n_i} \wr S_{m_i}$ and the quotient Q is isomorphic to $C_2 \wr S_{m_i} = (C_2)^{m_i} \rtimes S_{m_i}$. Then $(C_2)^{m_i}$ can be viewed as the natural permutation module for S_{m_i} over \mathbb{F}_2 . If U_2 denotes the S_{m_i} -submodule of $(C_2)^{m_i}$ defined in Lemma 2.4, then U_2 is a normal subgroup of Q and the quotient is isomorphic to $C_2 \times S_{m_i}$. Now A_{m_i} is a normal subgroup of $C_2 \times S_{m_i}$ and the quotient is isomorphic to $C_2 \times C_2$. Therefore we have proved that the wreath product $S_{n_i} \wr S_{m_i}$ has a normal subgroup N_i such that the quotient is isomorphic to $C_2 \times C_2$. Now, for $i \in \{1, \dots, u\}$, the subgroup A_{l_i} normal in S_{l_i} and the quotient is isomorphic to C_2 . Therefore the subgroup

$$N = N_1 \times \dots \times N_k \times A_{l_1} \times \dots \times A_{l_u}$$

is a normal subgroup of W such that W/N is isomorphic to $(C_2)^{2k+u}$. If W can be generated by fewer than $2k + u$ elements, then so can $W/N \cong (C_2)^{2k+u}$. However, the smallest generating set of $(C_2)^{2k+u}$ has $2k + u$ elements, and so the assertion is verified.

Next we show that W can indeed be generated by $2k + u$ elements. Set

$$W_1 = (S_{n_1} \wr S_{m_1}) \times \dots \times (S_{n_k} \wr S_{m_k}) \quad \text{and} \quad W_2 = S_{l_1} \times \dots \times S_{l_u}.$$

Then $W = W_1 \times W_2$. Since W_1 is the direct product of k groups each of which is generated by two elements (Theorem 2.3), we obtain that W_1 can be generated by $2k$ elements. For $u = 0$ the theorem is thus proved.

Suppose that $u = 1$. If $k = 0$ then S_{l_1} can be generated by 2 elements and there is nothing to prove. Suppose that $k \geq 1$ and consider the group $H = (S_{n_k} \wr S_{m_k}) \times S_{l_1}$. By the argument in the previous paragraph, it suffices to show that H is generated by 3 elements. Let x and y be the generators of $S_{n_k} \wr S_{m_k}$ given in Theorem 2.3. Set $u = (x, \text{id})$, $v = (y, (1, 2, \dots, l_1))$, and $w = (\text{id}, (1, 2))$. Then $u, v, w \in H$ and we claim that $H = \langle u, v, w \rangle$. Since the first components of u, v, w generate $S_{n_k} \wr S_{m_k}$ and the second components generate S_{l_1} , we have that $\langle u, v, w \rangle$ is a subdirect subgroup of $H = (S_{n_k} \wr S_{m_k}) \times S_{l_1}$. Set $N = \langle u, v, w \rangle \cap S_{l_1}$. For each $u_2 \in S_{l_1}$ there is some $u_1 \in S_{n_k} \wr S_{m_k}$ such that $(u_1, u_2) \in \langle u, v, w \rangle$. If $n \in N$ then $(\text{id}, n)^{(u_1, u_2)} = (\text{id}, n^{u_2})$ is an element of $N = \langle u, v, w \rangle \cap S_{l_1}$, and this shows that N is a normal subgroup of S_{l_1} . As $(1, 2) \in N$ and no proper normal subgroup of S_{l_1} contains the transposition $(1, 2)$, we find that $N = S_{l_1}$, and, in turn, that $S_{l_1} \leq \langle u, v, w \rangle$. As $\langle u, v, w \rangle$ is subdirect, we also obtain $S_{n_k} \wr S_{m_k} \leq \langle u, v, w \rangle$, and so $H = \langle u, v, w \rangle$. Thus shows that H is generated by three elements, and so W is generated by $2k + 1$ elements, as required.

Suppose now that $u \geq 2$. In this case, as W_1 is generated by $2k$ elements, we only need to show that W_2 is generated by u elements. Let $i \in \{1, \dots, u\}$. If l_i is even, then set $z_i = (2, \dots, l_i)$ otherwise set $z_i = (1, \dots, l_i)$. Therefore z_i is always a cycle of odd length such that $S_{l_i} = \langle (1, 2), z_i \rangle$. For $i \in \{1, \dots, u-1\}$ define

$$w_i = (\text{id}, \dots, \text{id}, \overset{i\text{-th component}}{(1, 2)}, \overset{(i+1)\text{-th component}}{z_{i+1}^2}, \text{id}, \dots, \text{id}) \in W_2$$

and also define

$$w_u = (z_1, \text{id}, \dots, \text{id}, (1, 2)) \in W_2.$$

We claim that $W_2 = \langle w_1, \dots, w_u \rangle$. Let o_i denote the order of z_i . As o_i is odd, all but the i -th component of $w_i^{o_i}$ is trivial, and the i -th component is $(1, 2)$. If $i \in \{1, \dots, u-1\}$, then in w_i^2 , all but the $(i+1)$ -th component is trivial, and the $(i+1)$ -th component is z_{i+1}^2 . Similarly, in w_u^2 all but the first component is trivial, and the first component is z_1^2 . Therefore, for $i \in \{1, \dots, u\}$, we obtain that the elements $(1, 2)$, $z_i^2 \in S_{l_i}$ are contained in $\langle w_1, \dots, w_u \rangle$. Since, the order of z_i is odd, z_i^2 is a cycle of the same length as z_i permuting the same points. Therefore $\langle (1, 2), z_i^2 \rangle = S_{l_i}$, which shows that $S_{l_i} \leq \langle w_1, \dots, w_u \rangle$. Since this is true for all i , we obtain that $W_2 \leq \langle w_1, \dots, w_u \rangle$, and the proof is complete. \square

Corollary 2.6. *Let \mathcal{P} be a partition on X , such that \mathcal{P} has exactly $m_i \geq 2$ blocks of size $n_i \geq 2$, $i = 1, \dots, p$, blocks of unique sizes l_1, \dots, l_q , where $l_i \geq 2$, and t singleton blocks (where p, q, t might be 0). If $|S(X, \mathcal{P})| \geq 3$ then the rank of $S(X, \mathcal{P})$ is*

$$\max\{2, 2p + q + g(t)\},$$

where $g(0) = g(1) = 0$ and $g(t) = 1$ for $t \geq 2$.

Proof. If $t = 0$ or $t = 1$ then $S(X, \mathcal{P})$ is isomorphic to

$$(S_{n_1} \wr S_{m_1}) \times \dots \times (S_{n_p} \wr S_{m_p}) \times S_{l_1} \times \dots \times S_{l_q}$$

and we may take $k = p$ and $u = q$ in Theorem 2.5. If $t \geq 2$ then $S(X, \mathcal{P})$ is isomorphic to

$$(S_{n_1} \wr S_{m_1}) \times \dots \times (S_{n_p} \wr S_{m_p}) \times S_{l_1} \times \dots \times S_{l_q} \times S_t,$$

and the statement follows from Theorem 2.5 with $k = p$, $u = q + 1$. \square

3. THE RELATIVE RANK OF $T(X, \mathcal{P})$ MODULO $\Sigma(X, \mathcal{P})$

Let \mathcal{A} denote the collection of those $f \in T(X, \mathcal{P})$ such that

- (i) $f|_{P_i}$ is injective for all i ;
- (ii) $|\text{im}(\bar{f})| = n - 1$;

Note that, by (i), if $(P_i)f \subseteq P_j$ and $|P_i| \neq |P_j|$, then $|P_i| < |P_j|$.

Lemma 3.1. *Let $f, g, a \in T(X, \mathcal{P})$ be arbitrary. Then the following hold:*

- (i) if $f \in \mathcal{A}$, $a \in \Sigma(X, \mathcal{P})$, and $f = ag$, then $a \in S(X, \mathcal{P})$;
- (ii) if $f \in \mathcal{A}$, $g \notin \Sigma(X, \mathcal{P})$, and $f = ga$, then $g \in \mathcal{A}$.
- (iii) if $f, g \in \mathcal{A}$ and $\bar{f} = \overline{ga}$, then there exist unique $i, j \in \{1, \dots, n\}$ such that $i \neq j$ and $(i)\bar{f} = (j)\bar{f} = (i)\bar{g} = (j)\bar{g}$;

Proof. (i). Since $\ker(a) \subseteq \ker(f)$ and f is injective on every $P_i \in \mathcal{P}$, it follows that a is injective on every $P_i \in \mathcal{P}$. But $a \in \Sigma(X, \mathcal{P})$ and so \bar{a} is a permutation. Thus a is a permutation, i.e. $a \in S(X, \mathcal{P})$.

(ii). As in the previous case, $\ker(g) \subseteq \ker(f)$, and since f is injective on every part of \mathcal{P} , it follows that g is too. Since $g \notin \Sigma(X, \mathcal{P})$, $|\text{im}(\bar{g})| \leq n - 1$. If $|\text{im}(\bar{g})| < n - 1$, then $|\text{im}(\overline{ga})| < n - 1 = |\text{im}(\bar{f})|$, a contradiction. So $|\text{im}(\bar{g})| = n - 1$, and $g \in \mathcal{A}$.

(iii). Similar to the previous cases, $\bar{f} = \overline{ga}$ implies that $\ker(\bar{g}) \subseteq \ker(\bar{f})$. But $f, g \in \mathcal{A}$, which implies that $|\text{im}(f)| = |\text{im}(g)| = n - 1$ and hence $\ker(\bar{f}) = \ker(\bar{g})$. \square

Lemma 3.2. *Let $U \subseteq T(X, \mathcal{P}) \setminus \Sigma(X, \mathcal{P})$ be such that $T(X, \mathcal{P}) = \langle \Sigma(X, \mathcal{P}), U \rangle$. Then for all distinct $i, j \in \{1, \dots, n\}$ there exist $f \in U \cap \mathcal{A}$ and distinct $k, l \in \{1, \dots, n\}$ such that $(k)\bar{f} = (l)\bar{f}$ and $|P_i| = |P_k|$ and $|P_j| = |P_l|$.*

Proof. Let $i, j \in \{1, \dots, n\}$ be arbitrary. Then there exists $f \in \mathcal{A}$ such that $(i)\bar{f} = (j)\bar{f}$. By assumption, $f \in \langle \Sigma(X, \mathcal{P}), U \rangle$ and so

$$f = s_1 a_1 s_2 a_2 \dots s_r a_n s_{r+1} \quad \text{for some } s_i \in \Sigma(X, \mathcal{P}), a_i \in U.$$

By Lemma 3.1(i), $s_1 \in S(X, \mathcal{P})$ and so $s_1^{-1}f \in \mathcal{A}$. If $k = (i)\overline{s_1}$ and $l = (j)\overline{s_1}$, then $(k)\overline{s_1^{-1}f} = (l)\overline{s_1^{-1}f}$. Since $s_1 \in S(X, \mathcal{P})$, it follows that $|P_k| = |P_i|$ and $|P_l| = |P_j|$. By Lemma 3.1(iii), $a_1 \in \mathcal{A}$ and thus, by Lemma 3.1(ii), $(k)\overline{a_1} = (l)\overline{a_1}$, as required. \square

We have everything we need to prove the main result of this section.

Theorem 3.3. *Let X be any finite set, \mathcal{P} a partition of X , s the number of distinct values $|P_i|$ and r the number of distinct values $|P_i|$ such that there are $i \neq j$ with $|P_i| = |P_j|$. Then $\text{rank}(T(X, \mathcal{P}) : \Sigma(X, \mathcal{P})) = \binom{s}{2} + r$.*

Proof. By Lemma 3.2, $\text{rank}(T(X, \mathcal{P}) : \Sigma(X, \mathcal{P})) \geq \binom{s}{2} + r$.

For the converse direction, given positive integers $p \leq q$, let $\mathcal{A}_{p,q}$ be the set of all $f \in \mathcal{A}$ such that $|P_i| = p, |P_j| = q$ where i, j are the members of the unique non-singleton class of $\ker \bar{f}$. It is clear that the non-empty $\mathcal{A}_{p,q}$ form a partition of \mathcal{A} with $\binom{s}{2} + r$ parts.

Let $U^{\mathcal{P}}$ be a set of representatives $f_{p,q}$ for the elements of this partition. We claim that $U^{\mathcal{P}}$ generates $T(X, \mathcal{P})$ over $\Sigma(X, \mathcal{P})$. We will first show that a particular set of functions can be generated from $U^{\mathcal{P}} \cup \Sigma(X, \mathcal{P})$.

For each $i < j$, and ϕ an injection from P_i to P_j , let $f_{i,j,\phi} \in T(X, \mathcal{P})$ be the function that agrees with ϕ on P_i and is the identity everywhere else. By Lemma 2.2, $f_{i,j,\phi} \in S(X, \mathcal{P})f_{|P_i|,|P_j|}S(X, \mathcal{P})$, and hence in $\langle \Sigma(X, \mathcal{P}) \cup U^{\mathcal{P}} \rangle$.

Let $f \in T(X, \mathcal{P})$. We will show that f can be generated from $\Sigma(X, \mathcal{P})$ and the $f_{i,j,\phi}$ using induction on the number of blocks in a partition \mathcal{Q} of an arbitrary finite set X' . The base case when there is only one block is trivial, since in this case $\Sigma(X', \mathcal{Q}) = T(X', \mathcal{Q})$. Our induction assumption is that $T(X', \mathcal{Q})$ is generated by $U^{\mathcal{Q}}$ and $\Sigma(X', \mathcal{Q})$ when Y is any finite set and \mathcal{Q} is any partition of Y with fewer than $n \in \mathbb{N}, n > 1$, blocks.

We construct several functions that are generated by $\Sigma(X, \mathcal{P})$ and the $f_{i,j,\phi}$ until we are able to use our inductive hypothesis.

Let $l = (n)\bar{f}$, and $i_1, \dots, i_k = n$ be the elements of the \bar{f} -kernel class of n . Let $D = P_{i_1} \cup \dots \cup P_{i_k}$.

Choose an injective function h from P_l to P_n . This is possible as $|P_n| \geq |P_l|$. Moreover for all j choose an injective function h_j from $\text{im}(f|_{P_{i_j}})$ to P_{i_j} . Such h_l exist, as the image $\text{im}(f|_{P_{i_j}})$ is not larger than the domain P_{i_j} .

Let e be the function for which $e|_D$ maps $y \in P_{i_j}$ to $((y)f)h_j$ and $e|_{X \setminus D}$ is the identity. Then \bar{e} is the identity, and hence $e \in \Sigma(X, \mathcal{P})$.

For each j , let ϕ_j be a function from P_{i_j} to P_n defined in the following way. For $x \in \text{im}(h_j)$, by construction there exists a $y \in P_{i_j}$ such that $x = ((y)f)h_j$. In this case set $(x)\phi_j = ((y)f)h$.

We have to show that this definition does not depend on the choice of y . So let $((y_1)f)h_j = ((y_2)f)h_j$ for some $y_1, y_2 \in P_{i_j}$. As h_j is an injection defined on the image of $f|_{P_{i_j}}$, we have that $(y_1)f = (y_2)f$, and hence $((y_1)f)h = ((y_2)f)h$ and so ϕ_j is well-defined for every $x \in \text{im}(h_j)$.

Now let $x_1 \neq x_2, x_1, x_2 \in \text{im}(h_j)$, say $((y_1)f)h_j = x_1$ and $((y_2)f)h_j = x_2$. Then $(y_1)f \neq (y_2)f$ and as h is injective $(x_1)\phi_j = ((y_1)f)h \neq ((y_2)f)h = (x_2)\phi_j$. Hence ϕ_j is injective on $\text{im}(h_j)$. Now extend ϕ_j arbitrary to all of P_{i_j} , subject to ϕ_j being an injection. Such ϕ_j exists as P_n is the block of largest size. Let

$$g = e f_{i_1, n, \phi_1} \dots f_{i_k, n, \phi_k}.$$

It is straightforward to check that g satisfies the following properties:

- (1) $g|_{X \setminus D}$ is the identity,
- (2) $(\{i_1, \dots, i_k\})\bar{g} = \{n\}$, and \bar{g} is the identity otherwise,
- (3) $g|_D$ and $f|_D$ have the same kernel,

- (4) if $x \in D$, then $(x)g = ((x)f)h$.

Next let v be a function constructed as follows: v maps P_n to P_l so that any $x \in \text{im}(h)$ is mapped to xh^{-1} and is arbitrary otherwise (recall that h was injective). For $j = l, \dots, n-1$, v maps P_j injectively into P_{j+1} and is the identity everywhere else. Clearly such v exists and is an element of $\Sigma(X, \rho)$. Let $g' = gv$. We claim that g' has the following properties:

- (1) $g'|_{X \setminus D}$ is injective,
- (2) $(\{i_1, \dots, i_k\})\bar{g}' = \{l\}$, and \bar{g}' maps all other values injectively to values different from l ,
- (3) if $x \in D$, then $(x)g' = (x)f$,
- (4) $\ker g' \subseteq \ker f$,
- (5) if $(x)g'$ and $(y)g'$ are in the same part of \mathcal{P} , then $(x)f$ and $(y)f$ are in the same part of \mathcal{P} .

The first two properties follow from the corresponding results for g . For the third, let $x \in D$, then $(x)g' = ((x)g)v = (((x)f)h)v = (x)f$, and so g' agrees with f on D . The fourth assertion follows from the first and third, and the final one from the second and fourth one.

We will next construct a new function h' . If $x \in \text{im}(g')$, say $x = (y)g'$, then set $(x)h' = (y)f$. As $\ker g' \subseteq \ker f$, this function is well-defined. By the last property of g' , this partial assignment preserves \mathcal{P} . If $x \notin \text{im}(g')$, $x \in P_i$ and there is a $y \in P_i \cap \text{im}(g')$, then let $(x)h' = x$ if $(y)h' \in P_i$, or otherwise be an arbitrary element of the part of $(y)h'$. Once again by the last property of g' , the condition is well defined and the assignment so far continues to preserve \mathcal{P} . Finally if i is such that $P_i \cap \text{im}(g')$ is empty, then pick a $P_j \in \mathcal{P}$, with $j \neq l$, and let h' map all of P_i into P_j in an arbitrary way.

The function h' has the following properties:

- (1) $h' \in T(X, \mathcal{P})$
- (2) $g'h' = f$
- (3) $(\{l\})\bar{h}'^{-1} = (\{l\})$.
- (4) h' is the identity on P_l .

The first two properties and the fact that $(P_l)h' \subseteq P_l$ follow directly from the construction of h' . Conversely let $x \in P_i$ with $i \neq l$. If $\text{im}(g') \cap P_i$ is empty, then the above construction maps x into a part different from P_l . If there is an element in $\text{im}(g') \cap P_i$, which we may assume w.l.o.g. to be x , let $x = (y)g'$. Then $y \notin \{P_{i_1}, \dots, P_{i_k}\}$ by the second property of g' . But then $(x)h' = (y)f$ cannot be in P_l as $(\{l\})\bar{f}^{-1} = \{i_1, \dots, i_k\}$. So $(\{l\})\bar{h}'^{-1} = (\{l\})$.

For the last property, let $x \in P_l \cap \text{im}(g')$, say $(y)g' = x$. By property (2) of g' , $x \in D$, and hence, by property (3) of g' , $(x)h' = (y)f = (y)g' = x$. As h' maps the elements of $P_l \cap \text{im}(g')$ into P_l , it maps $P_l \setminus \text{im}(g')$ identical by its definition.

Now let $X' = X \setminus P_l$ and \mathcal{Q} be the partition of X' given by $\mathcal{P} \setminus \{P_l\}$. Let $f' = h'|_{X \setminus P_l}$. As $(\{l\})\bar{h}'^{-1} = (\{l\})$, $f' \in T(X', \mathcal{Q})$. By the induction assumption, $f' = g'_1 \dots g'_j$, where the g'_i are either from $\Sigma(X', \mathcal{Q})$, or of the form $f'_{s,t,\phi}$. Extend each g'_i to a function g_i in $T(X, \mathcal{P})$, by letting $g_i|_{P_l}$ be the identity. It is clear that the g_i are either in $\Sigma(X, \mathcal{P})$ or are of the form $f_{s,t,\phi}$. Moreover, as h' is the identity on P_l , $h' = g_1 \dots g_j$. Hence $h' \in \langle \Sigma(X, \mathcal{P}) \cup U^{\mathcal{P}} \rangle$, and so $f = g'h' \in \langle \Sigma(X, \mathcal{P}) \cup U^{\mathcal{P}} \rangle$, as required. \square

Corollary 3.4. *Let \mathcal{P} be a partition on X , such that \mathcal{P} has exactly $m_i \geq 2$ blocks of size $n_i \geq 2$, $i = 1, \dots, p$, blocks of unique sizes l_1, \dots, l_q , where $l_i \geq 2$, and t singleton blocks (where p, q, t might be 0). Then the rank of $\Sigma(X, \mathcal{P})$ modulo $S(X, \mathcal{P})$ is*

$$\binom{p+q}{2} + p + h(p, q, t)$$

where $h(p, q, 0) = 0$, $h(p, q, 1) = p + q$ and $h(p, q, t) = p + q + 1$ if $t \geq 2$.

Proof. If $t = 0$, we may take $s = p + q$ and $r = p$ in Theorem 3.3.

If $t = 1$, with $s = p + q + 1$ and $r = p$, we get that the rank of $\Sigma(X, \mathcal{P})$ modulo $S(X, \mathcal{P})$ equals

$$\binom{p+q+1}{2} + p = \binom{p+q}{2} + (p+q) + p.$$

Finally, if $t = 2$, the result follows analog with $s = p + q + 1$, $r = p + 1$. \square

4. THE RANK OF $\Sigma(X, \mathcal{P})$ OVER $S(X, \mathcal{P})$

As in the previous sections, suppose $\mathcal{P} = \{P_1, \dots, P_n\}$, with $|P_i| \leq |P_{i+1}|$ and let $l_1 < l_2 < \dots < l_r$ be the distinct sizes of blocks in \mathcal{P} .

For $i \leq r-1$, let \mathcal{B}_i be the set of all mappings $f \in \Sigma(X, \mathcal{P})$ such that there are $P_j, P_{j'}, P_k, P_{k'}$ with $|P_j| = l_i = |P_{j'}|$, $|P_k| = l_{i+1} = |P_{k'}|$, such that

- (1) f maps P_j injectively to P_k
- (2) f maps $P_{k'}$ surjectively onto $P_{j'}$
- (3) f maps every other block bijectively to a block of the same size.

We do not exclude the possibility that $j = j'$ or $k = k'$. Clearly, \mathcal{B}_i is non-empty for all $i \leq r-1$, and any element of \mathcal{B}_i has image size $|X| - l_{i+1} + l_i$.

Lemma 4.1. *If $\langle S(X, \mathcal{P}), U \rangle = \Sigma(X, \mathcal{P})$ for some $U \subseteq \Sigma(X, \mathcal{P})$, then $\mathcal{B}_i \cap U \neq \emptyset$ for every $i \leq r-1$.*

Proof. Let $i \in \{1, \dots, r-1\}$ be arbitrary. Then there is an $f \in \mathcal{B}_i$ such that $\bar{f} = (j \ k)$ with $j < k$ and j and k are minimal and maximal among those indices of blocks with sizes equal to $|P_j| = l_i$ and $|P_k| = l_{i+1}$, respectively. By assumption, there exist $g_1, \dots, g_m \in S(X, \mathcal{P}) \cup U$ such that $f = g_1 \dots g_m$ and hence $\bar{f} = \bar{g}_1 \dots \bar{g}_m$. Since $j\bar{f} = k$, it follows that $(j)\bar{g}_1 \dots \bar{g}_m = k$. It follows, since $f|_{P_j}$ is injective, and by the minimality of j , that $(j)\bar{g}_1, (j)\bar{g}_1\bar{g}_2, \dots, (j)\bar{g}_1\bar{g}_2 \dots \bar{g}_m \geq j$.

Let u be the least value for which $|P_j| < |P_{(j)\bar{g}_1 \dots \bar{g}_u}|$, and let $t = (j)\bar{g}_1 \dots \bar{g}_{u-1}$. Then at least $|P_{(t)\bar{g}_u}| - |P_t| = |P_{(t)\bar{g}_u}| - |P_j|$ elements of $P_{(t)\bar{g}_u}$ are not in the image of g_u . But f has image size $|X| - |P_k| + |P_j|$ and so $g_u|_{P_t}$ is injective, $|P_{(t)\bar{g}_u}| = |P_k|$, and $X \setminus P_{(t)\bar{g}_u}$ is contained in the image of g_u . It follows that \bar{g}_u is a permutation that maps every block other than P_t onto a block of equal or smaller size.

Since \bar{g}_u is a permutation, and there is exactly one block of \mathcal{P} mapped to a larger block, there is also exactly one block $P_{k'}$ which is mapped to a smaller block $P_{j'}$. Due to the restriction on the size of the image of f , it follows that $|P_{(t)\bar{g}_u}| = |P_k|$ and $|P_{j'}| = |P_j|$. As g_u maps every block other than P_t surjectively onto its image block, it follows that $g_u \in \mathcal{B}_i \cap U$. \square

For each $i \leq r$ let \mathcal{C}_i be the set of all $f \in \Sigma(X, \mathcal{P})$ such that

- (1) f maps each block to one of the same size (potentially itself);
- (2) there is one block of size l_i whose image under f has size $l_i - 1$;
- (3) f maps all other blocks injectively.

Clearly, any such f has image size $|X| - 1$, and \mathcal{C}_i is non-empty except when $i = 1$ and $l_1 = 1$.

Lemma 4.2. *If $\langle S(X, \mathcal{P}), U \rangle = \Sigma(X, \mathcal{P})$ for some $U \subseteq \Sigma(X, \mathcal{P})$, and $i \in \{1, \dots, r\}$ is such that either $i = 1$ and $l_1 \neq 1$ or $i \geq 2$ and $l_i - l_{i-1} \geq 2$, then $\mathcal{C}_i \cap U \neq \emptyset$.*

Proof. Let $f \in \mathcal{C}_i$ be arbitrary. Then $f = h_1 h_2 \dots h_m$ for some $h_1, h_2, \dots, h_m \in S(X, \mathcal{P}) \cup U$. As mentioned above, the image of f has size $|X| - 1$.

Let z be the smallest index for which there is a block P_k of size l_i that is not contained in the image of $h_1 \dots h_z$. Clearly, the image of h_z must contain $l_i - 1$ elements of P_k . Since $h_z \in \Sigma(X, \mathcal{P})$, it follows that \bar{h}_z is a permutation. We set $j = (k)\bar{h}_z^{-1}$.

We will show that $|P_j| = l_i$. By way of contradiction, assume that $|P_j| < l_i$. This is not possible for $i = 1$, and if $i \geq 2$ then our condition on i implies that $|P_j| < l_i - 2$. However in the latter case, there would be at least two elements of P_k that were not in the image of h_z , contradicting the assumption that f has image size $|X| - 1$. So $|P_j| \geq l_j$.

If $|P_j| > l_i$ then (as \bar{h}_z is a permutation on a finite set) there must be one other index j' such that $j'\bar{h}_z = k'$ with $|P_{j'}| < |P_{k'}|$ and $k \neq k'$. However, in this case $P_{k'}$ and P_k would not be contained in the image of h_z , once again contradicting the assumption on the image size of f .

We have shown that $|P_j| = l_i$. Note that h_z must map $X \setminus P_j$ bijectively to $X \setminus P_k$, once again by considering the size of the image of f . It follows that $h_z \in \mathcal{C}_i \cap U$. \square

We define

$$\mathcal{B} = \bigcup_{i=1}^{r-1} \mathcal{B}_i \quad \text{and} \quad \mathcal{C} = \bigcup_{i=1}^r \mathcal{C}_i.$$

Let $f \in S_n$. Then $g \in \Sigma(X, \mathcal{P})$ is said to be a *companion of f* if

- (1) $\bar{g} = f$;
- (2) if $|P_i| \leq |P_{(i)f}|$, then $g|_{P_i} : P_i \rightarrow P_{(i)f}$ is injective;
- (3) if $|P_i| \geq |P_{(i)f}|$, then $g|_{P_i} : P_i \rightarrow P_{(i)f}$ is surjective.

Lemma 4.3. *If $f \in \Sigma(X, \mathcal{P})$ and there is a companion for \bar{f} in $\langle S(X, \mathcal{P}), \mathcal{B} \rangle$, then $f \in \langle S(X, \mathcal{P}), \mathcal{B}, \mathcal{C} \rangle$.*

Proof. If $k \in \{1, \dots, n\}$ is such that $|P_k| = l_i > 1$ for some i , then there exists $t_k \in \mathcal{C}_i$ which is the identity outside P_k . It is well-known that for any finite set Y with at least two elements, every function on Y is a product of permutations and a fixed function with image size $|Y| - 1$. Therefore t_k and $S(X, \mathcal{P})$ generate every element of $T(X, \mathcal{P})$ which maps P_k to P_k and fixes $X \setminus P_k$. It follows that every $f \in \Sigma(X, \mathcal{P})$ such that \bar{f} is the identity belongs to $\langle S(X, \mathcal{P}), \mathcal{C} \rangle$.

Let $f \in \Sigma(X, \mathcal{P})$. Then by assumption there exists $g \in \langle S(X, \mathcal{P}), \mathcal{B} \rangle$ such that g is a companion for \bar{f} . From the preceding paragraph, there is an idempotent $e \in \langle S(X, \mathcal{P}), \mathcal{C} \rangle$ such that $\ker(e) = \ker(f)$. It follows that there exists $h \in S(X, \mathcal{P})$ such that $f = ehg \in \langle S(X, \mathcal{P}), \mathcal{B}, \mathcal{C} \rangle$. \square

Lemma 4.4. *If $f \in \Sigma(X, \mathcal{P})$, then there exists a companion for \bar{f} in $\langle S(X, \mathcal{P}), \mathcal{B} \rangle$.*

Proof. Since every permutation is a product of disjoint cycles, there exists a companion for $f \in \Sigma(X, \mathcal{P})$ in $\langle S(X, \mathcal{P}), \mathcal{B} \rangle$ if and only if there is a companion in $\langle S(X, \mathcal{P}), \mathcal{B} \rangle$ for every cycle in S_n .

For any $k \leq n - 1$ let $f_{(k \ k+1)}$ be such that $\bar{f}_{(k \ k+1)} = (k \ k+1)$, $f_{(k \ k+1)}|_{P_{k+1}}$ maps onto P_k , the image of $f_{(k \ k+1)}|_{P_k}$ is a section for the kernel of $f_{(k \ k+1)}|_{P_{k+1}}$, and $f_{(k \ k+1)}$ is the identity outside of $P_k \cup P_{k+1}$. Since $|P_{k+1}| = |P_k|$, it follows that f is injective on P_k and so $f_{(k \ k+1)}$ is a companion for $(k \ k+1)$. Moreover, $f_{(k \ k+1)}$ belongs to \mathcal{B} when $|P_k| < |P_{k+1}|$ and it belongs to $S(X, \mathcal{P})$ when $|P_k| = |P_{k+1}|$.

Suppose that $i < j$. Then it is straightforward to check that

$$f_{(i \ j)} = f_{(j-1 \ j)} f_{(j-2 \ j-1)} \cdots f_{(i+1 \ i+2)} f_{(i \ i+1)} f_{(i+1 \ i+2)} f_{(i+2 \ i+3)} \cdots f_{(j-2 \ j-1)} f_{(j-1 \ j)}$$

is a companion for $(i \ j)$.

We proceed by induction on the length k of a cycle. Suppose that for some k with $2 \leq k < n$, there exists a companion in $\langle S(X, \mathcal{P}), \mathcal{B} \rangle$ for every cycle of length at most k . Let $h = (x_1 \dots x_{k+1})$ and let $x_j = \min\{x_1, \dots, x_{k+1}\}$. Then

$$h = (x_{j+1} x_{j+2} \dots x_{k+1} x_1 \dots x_{j-1}) (x_j \ x_{j+1}).$$

By induction, there is a companion $h_1 \in \langle S(X, \mathcal{P}), \mathcal{B} \rangle$ for $(x_{j+1} x_{j+2} \dots x_{k+1} x_1 \dots x_{j-1})$. It follows that $|\text{im}(h_1|_{P_{x_{j-1}}})| = \min\{|P_{x_{j-1}}|, |P_{x_{j+1}}|\} \geq |P_{x_j}|$ by the minimality of x_j .

Let $g \in S(X, \mathcal{P})$ be such that g maps a subset of $\text{im}(h_1|_{P_{x_{j-1}}}) \subseteq P_{x_{j+1}}$ onto a section of the kernel of $f_{(x_j \ x_{j+1})}|_{P_{x_{j+1}}}$, and is the identity outside of $P_{x_{j+1}}$. Since $f_{(x_j \ x_{j+1})}|_{P_{x_{j+1}}}$ has $|P_{x_j}|$ kernel classes such g exists due to our estimate above. It follows that $h_1 g f_{(x_j \ x_{j+1})}$ is a companion for h . \square

The two previous results imply the following corollary.

Corollary 4.5. $\Sigma(X, \mathcal{P}) = \langle S(X, \mathcal{P}), \mathcal{B}, \mathcal{C} \rangle$.

Theorem 4.6. *Let U be a set that contains one representative from each \mathcal{B}_i , for $i \leq r - 1$, and one representative from each \mathcal{C}_i , for all i that satisfy the condition in the statement of Lemma 4.2 (i.e. either $i = 1$ and $l_1 \geq 2$ or $i \geq 2$ and $l_i - l_{i-1} \geq 2$). Then $\Sigma(X, \mathcal{P}) = \langle S(X, \mathcal{P}), U \rangle$.*

Proof. By Corollary 4.6, it suffices to show that $S(X, \mathcal{P}) \cup U$ generates \mathcal{B} and \mathcal{C} .

Considering \mathcal{C} , we will first show that for each P_x with $|P_x| > 1$, there exists an f_x such that $P_y f_x \subseteq P_y$ for all $y \leq n$, f_x has image size $n - 1$, and that P_x is the unique block that is not mapped injectively by f_x .

Let i be such that $l_i = |P_x|$. If either $i = 1$ (in which case $l_1 = |P_x| \geq 2$), or $l_i - l_{i-1} \geq 2$, then there exists $f_{x'} \in U \cap \mathcal{C}_i$, that is not injective on $P_{x'}$ with $|P_{x'}| = |P_x|$. But then $f_x \in S(X, \mathcal{P})f_{x'}S(X, \mathcal{P})$ by Lemma 2.2.

If $i \geq 2$ and $l_i - l_{i-1} = 1$, let h_{i-1} be the element in $\mathcal{B}_{i-1} \cap U$. Let f_{xy} be a mapping that maps P_x onto some P_y with $|P_y| = l_{i-1} = |P_x| - 1$, maps P_y injectively to P_x and is the identity everywhere else. By Lemma 2.2, we have that $f_{xy} \in S(X, \mathcal{P})h_{i-1}S(X, \mathcal{P})$. Now $f_x := f_{xy}^2$ can easily be checked to have the claimed properties.

Now for general $h \in \mathcal{C}_i$, $l_i \neq 1$, as otherwise \mathcal{C}_i would be empty. Choose a P_x with $|P_x| = l_i$, then $g \in S(X, \mathcal{P})f_xS(X, \mathcal{P})$ by Lemma 2.2. Hence $\mathcal{C} \subseteq \langle S(X, \mathcal{P}), U \rangle$, as required.

Now, for each z with $|P_z| = l_i \geq 2$, there is a function $f_z \in \langle \mathcal{C} \subseteq S(X, \mathcal{P}) \cup U \rangle$ that maps P_z to itself, is the identity everywhere else, and has an image that intersects P_z with size $l_i - 1$. Consider the subsemigroup Q_z of $T(X, \mathcal{P})$, consisting of all elements that map P_z into itself and are the identity outside of P_z . Q_z is clearly isomorphic to TP_z , the full transformation semigroup on P_z . This semigroup is generated by its units together with a transformation of rank $|P_z| - 1$. It follows that $Q_z \subseteq \langle (S(X, \mathcal{P}) \cap S_z) \cup \{f_z\} \rangle$ for every z (note that this also holds trivially if $|P_z| = 1$).

Now consider any element $h \in \mathcal{B}_i$, for $i \leq r - 1$, and let P_z be the unique part of \mathcal{P} that is not mapped injectively by h . There exists an $h' \in Q_z$ that has the same kernel classes on P_z as h . But both $h \in \mathcal{B}_i$ and $h' \in Q_z$ only have singleton kernel classes outside of P_z , and hence $h \in S(X, \mathcal{P})h'S(X, \mathcal{P})$ by Lemma 2.2. So $h \in \langle S(X, \mathcal{P}) \cup U \rangle$, as required. \square

Corollary 4.7. *Let \mathcal{P} be a partition on X , such that \mathcal{P} has exactly $m_i \geq 2$ blocks of size $n_i \geq 2$, $i = 1, \dots, p$, blocks of unique sizes l_1, \dots, l_q , where $l_i \geq 2$, and t singleton blocks (where p, q, t might be 0). Then*

$$\text{rank}(\Sigma(X, \mathcal{P}) : S(X, \mathcal{P})) = p + q + g'(t) - 1 + l$$

where

- $g'(0) = 0$ and $g'(t) = 1$ for $t \geq 1$,
- l is the number of values s for which \mathcal{P} has a block of size $s \geq 2$, but no block of size $s - 1$.

Proof. From Lemma 4.1, Lemma 4.2, and Theorem 4.6 it follows that $\text{rank}(\Sigma(X, \mathcal{P}) : S(X, \mathcal{P}))$ is one less than the number of distinct block sizes of \mathcal{P} plus the number of block sizes that satisfy the conditions mentioned in Lemma 4.2. The first of these numbers is $p + q + g'(t) - 1$ and the second is l . \square

5. PROBLEMS

Let X be a finite set, let \mathcal{P} be a partition and S be a section, that is, for every $P \in \mathcal{P}$ we have that $S \cap P$ is a singleton set. Given a set $Y \subseteq X$, we say that $f \in T(X)$ stabilizes Y if $Yf \subseteq Y$. Now consider the semigroup

$$T(X, \mathcal{P}, S) = \{f \in T(X) \mid f \text{ stabilizes } \mathcal{P} \text{ and } S\}.$$

This semigroup, in addition to the obvious similarities with $T(X, \mathcal{P})$, has many interesting properties:

- (1) both $T(X)$ and $PT(X)$, the semigroup of partial transformations on X , are examples of semigroups of this type; for instance, $T(X)$ is $T(X, \{\{x\} \mid x \in X\}, X)$ and $PT(X)$ is isomorphic (for an element $0 \notin X$) to $T(X \cup \{0\}, \{X \cup \{0\}\}, \{0\})$ (see [3, 4]).
- (2) Let $e^2 = e \in T(X)$; the centralizer of e in $T(X)$ is $C(e) = \{f \in T(X) \mid fe = ef\}$. Then $C(e) = T(X, \ker(e), Xe)$ (see [3, 4]). In this setting, $T(X)$ is the centralizer of the identity and $PT(X)$ is the centralizer of a constant map.
- (3) $T(X, \mathcal{P}, S)$ is regular if and only if either
 - (a) no part in \mathcal{P} has more than 2 elements; or
 - (b) at most one of the parts in \mathcal{P} has size larger than 1.
(See [4].)
- (4) The singular elements of a regular $C(e)$ are generated by idempotents if and only if e is the identity or a constant (see [1]).

- (5) Taking into account that in the Cayley table of a semigroup, each column (seen as map) is contained in the centralizer of each row, the semigroups $T(X, \mathcal{P}, S)$ have some surprising consequences in equational logic (see [2]).

Therefore given the importance of $T(X, \mathcal{P}, S)$ and its similarities with the semigroups $T(X, \mathcal{P})$, the following problems are very natural.

Problem 5.1. *Find the rank $T(X, \mathcal{P}, S)$, when \mathcal{P} is uniform and $T(X, \mathcal{P}, S)$ is regular. (Given the results above, this amounts to find the rank of $T(X, \mathcal{P}, S)$ when all parts in \mathcal{P} have exactly two elements.)*

The previous problem is a partial analogous of the main result in [7]. The full analogous would be the following.

Problem 5.2. *Find the rank $T(X, \mathcal{P}, S)$, when \mathcal{P} is uniform.*

Given the importance of regular semigroups in semigroup theory the following problem is also natural.

Problem 5.3. *Find the rank of the regular semigroups $T(X, \mathcal{P}, S)$. (The solution of this problem requires the solution of Problem 5.1.)*

Obviously, the ultimate goal of this sequence of problems would be the solution of the problem analogous to the main problem solved in this paper.

Problem 5.4. *Find the rank of the regular semigroups $T(X, \mathcal{P}, S)$.*

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